ANALYSIS I Lecture 5

Evening sessions.

Séances du soir, MATH-101 analyse I pour sections d'ingénierie Automne 2025-2026. Semaines 4 à 13 comprises			
Jour	Horraire	Salle	Cours représentés
Lundi	17h30-19h00	CM 1 105	SV, SIE/GC/SC, MT, IN Strütt, Mila, Mounford, Lachowsa F, A, D, E
Mardi	17h30-19h00	BS 170	Inversée, EL/MX/CGC, IN, EN pl, G, E, en Friedli, Basterrechea, Lachowska, Monin
Mercredi	17h30-19h00	CO 122	GM, MT C, D Friedli, Mountford
Jeudi	18h15-19h45	MA B1 11	SIE/GC/SC, GM, EL/MX/CGC A, C, G Mila, Friedli, Basterrechea

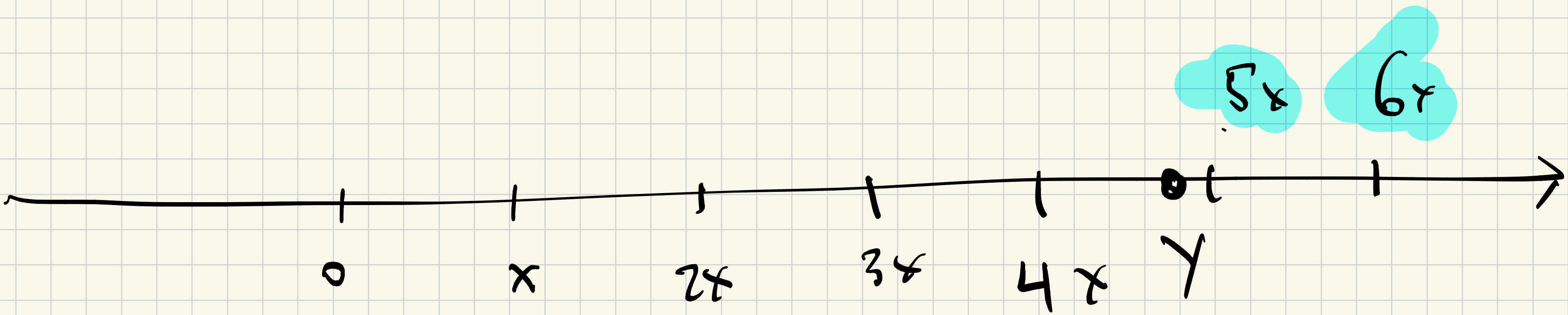
Principe : Vous êtes bienvenu·es à n'importe quelle séance. Les cours représentés sont donnés à titre d'information si vous avez besoin de parler à un·e assistant·e de votre cours pour une question spécifique.

Last time

Archemidean property

For any $x > 0$ and any $y \in \mathbb{R}$

$\exists\, n \in \mathbb{N}^*$ s.t. $n \cdot x > y$.



Floor / Ceiling functions $\lfloor x \rfloor$ $\lceil x \rceil$

Largest integer $\leq x$

Smallest integer $\geq x$

Integer / fractional parts $[x]$, $\{x\}$

for decimal expression:

$[-5.6] = -5$

$\{-5.6\} = -0.6$

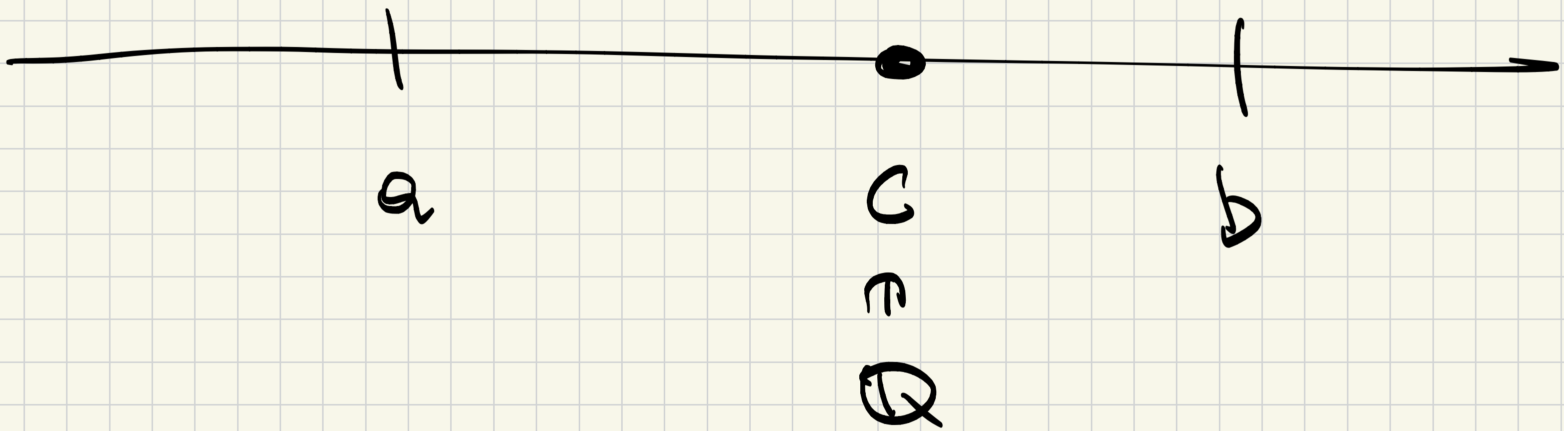
$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0 \end{cases}$$

$\{x\} = x - [x]$

Density of $\mathbb{Q}$ in $\mathbb{R}$:

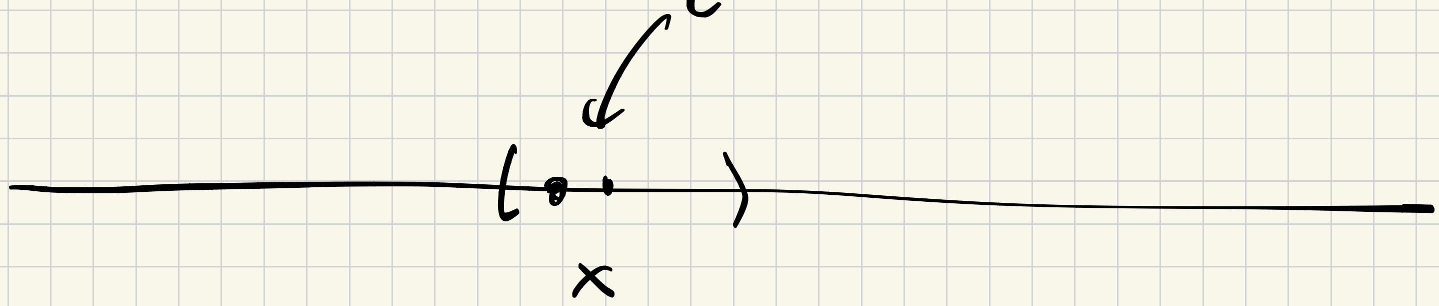
$\forall\ a < b \in \mathbb{R} \quad \exists\ c \in \mathbb{Q}$ with

$$a < c < b$$



In particular, for any $x \in \mathbb{R}$ we can find a rational number as close as possible to x.

$c \in (x-\varepsilon,\ x+\varepsilon)$



Similarly irrationals are dense in $\mathbb{R}$

$$(\mathbb{R} \setminus \mathbb{Q})$$

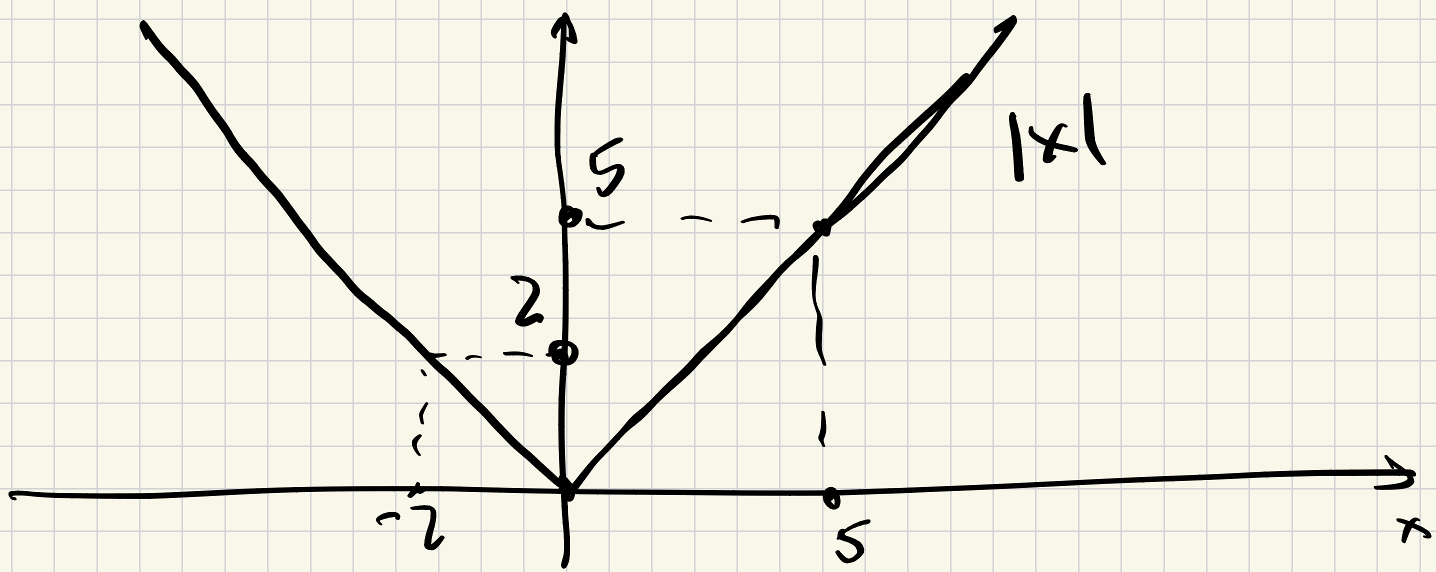
Theorem for any $a < b \in \mathbb{R}$

$\exists \ c \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < c < b$.

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In particular, $|x| \geq 0$



Some properties

- $|x \cdot y| = |x| \cdot |y|$

- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Prove this by case analysis:

$x \geq 0, y \geq 0$; $x \geq 0 \;\; y < 0$; $x < 0, y \geq 0$; $x < 0, y < 0$

Example

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\sqrt{(\underline{-5})^2} = \sqrt{25} = 5$$

$$\sqrt{} = \sqrt{}$$

Triangle inequality

Proposition

- $|x+y| \leq |x|+|y|$

Moreover we have equality iff x and y have the same sign

and we have strict equality otherwise

Proof $\Rightarrow$ Direct computation considering the same cases.

$$|(-5) + (-3)| = |-8| = 8 = 5+3$$
$$= |-5| + |-3|$$

$$|(-5) + 3| = |-2| = 2 < 8 = |-5| + |3|$$

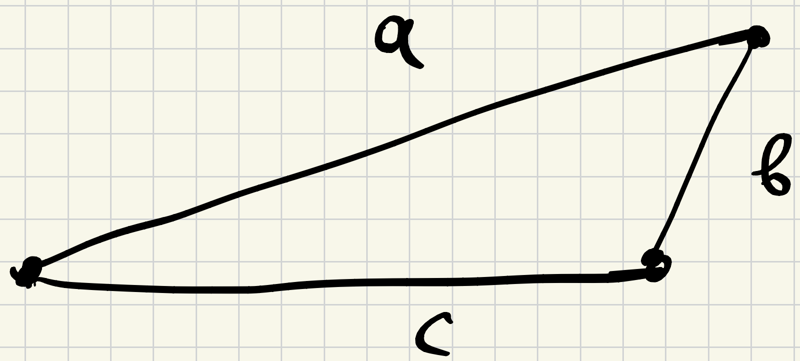
Why triangle inequality?

for which $a, b, c \in \mathbb{R}_{>0}$

positive real numbers

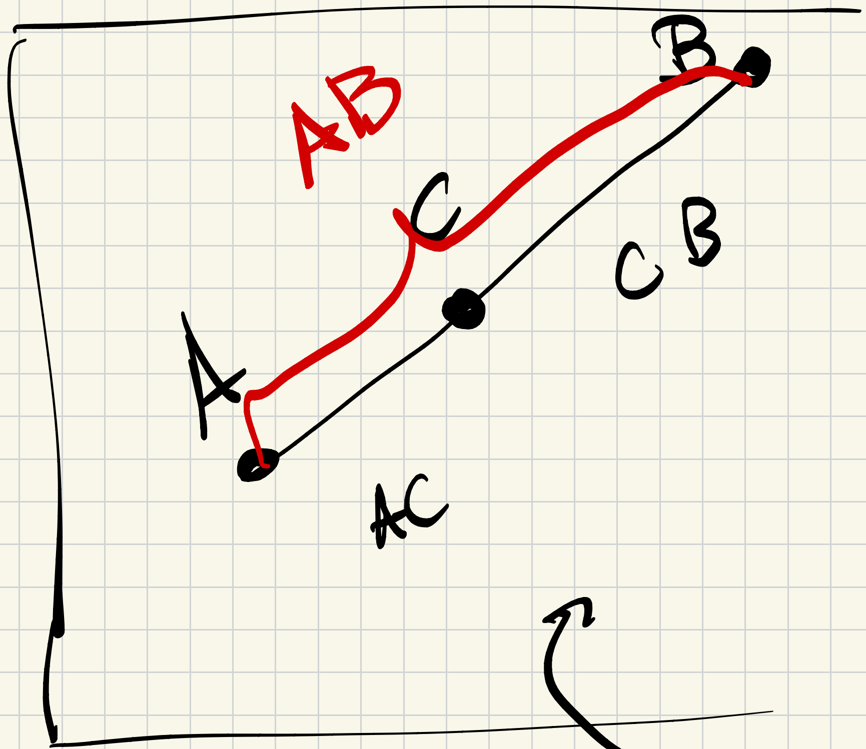
there exists a triangle in $\mathbb{R}^2$

with side lengths a, b, c



Fact Such triangle exists iff:

$$\boxed{\begin{aligned} a &< b+c \\ b &< a+c \\ c &< a+b \end{aligned}}$$



Remark We can include a, b, c satisfying non-strict inequalities if you allow degenerate triangles

For absolute values:

$$|x+y| \leq |x| + |y|$$

We also have:

$$|x| \leq |x+y| + |y|$$

$$|y| \leq |x+y| + |x|$$

Notice $|y| = |-y|$

and $(x+y) + (-y) = x$

$$= \underbrace{|x+y|}_{a} + \underbrace{|-y|}_{b} \geq |a+b|$$

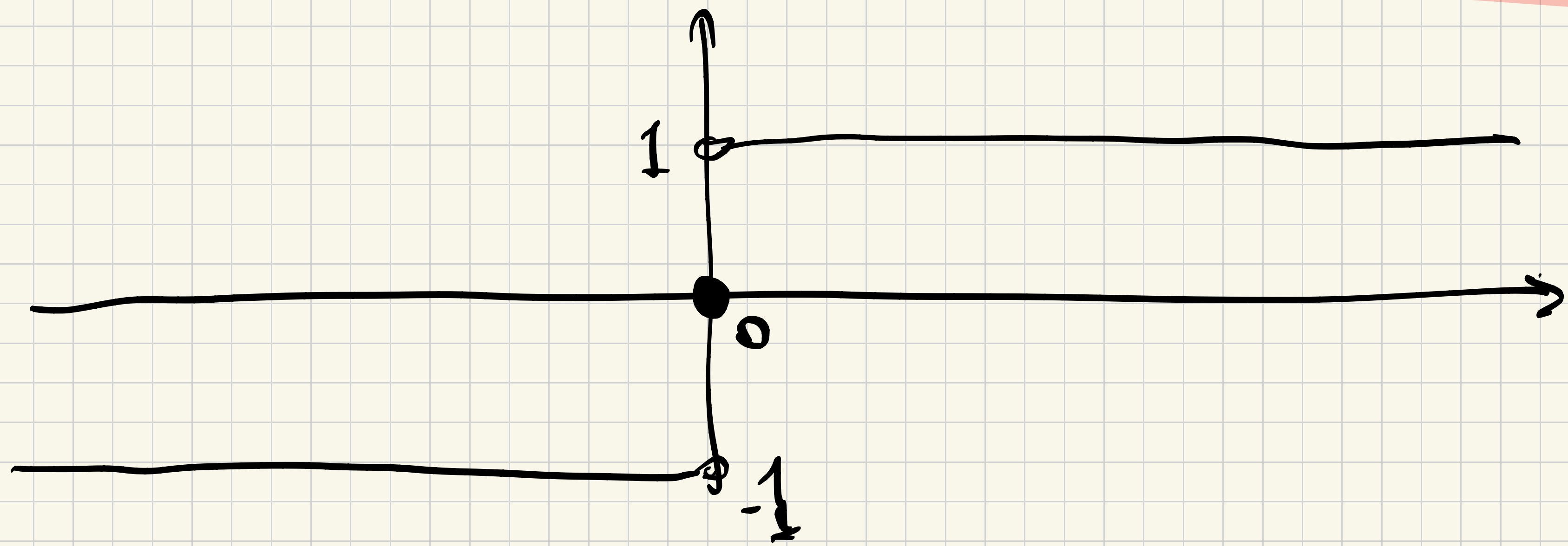
$$|a+b| = x$$

Sign function

For $x \neq 0$:

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ 0 & \text{if } x = 0 \\ -1 & x < 0 \end{cases}$$

Exercise Show that $\operatorname{sgn}(x) = \dfrac{x}{|x|}$ if $x \neq 0$



Complex numbers

Motivation $\sqrt{2}$ solution to

$x^2 = 2$ and is

not rational but

it real.

There are still equations which can not be solved in $\mathbb{R}$: $x^2 = -1$

Indeed, $x^2 \geq 0$ but $-1 < 0$

for $x \in \mathbb{R}$

More generally, maybe we want to be able to solve all polynomial equations.

It would be really nice if we have some numbers $\mathbb{C}$ such that any polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

has a root $z \in \mathbb{C}$.

i.e. z is a solution of $f(x) = 0$

Turns out such set $\mathbb{C}$ exists and is obtained from $\mathbb{R}$ by "adding" just one extra number

$$i = \sqrt{-1}$$

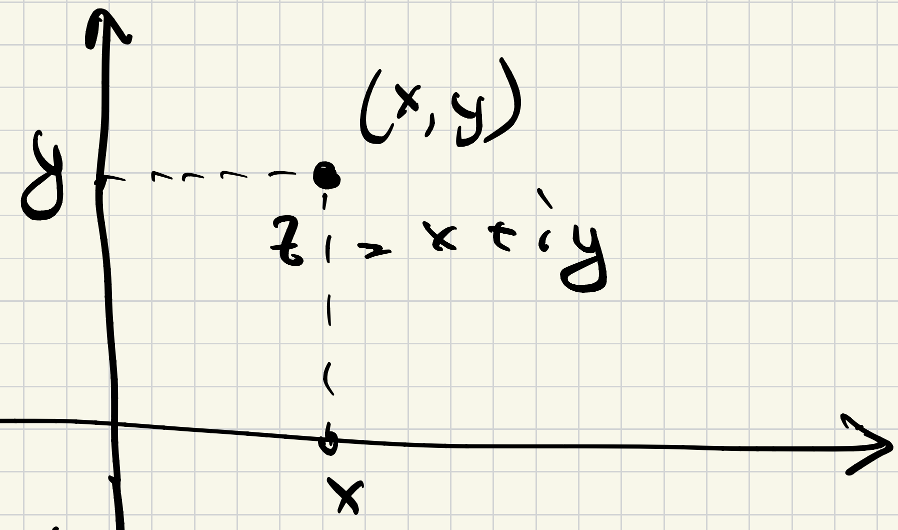
Means that by definition $i^2 = -1$.

imaginary unit

Definition of complex numbers

$$\mathbb{C} = \{ \underline{x + i \cdot y} \mid x, y \in \mathbb{R} \}$$

Here i is a symbol.



In particular,

Naively a complex number is a pair of real numbers.

But complex numbers form a field!

you can multiply, add, substract, divide

+ Usual properties of arithmetic operations! $a \cdot b = b \cdot a$; $a + b = b + a$;

$a(b+c) = ab + ac$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c$; similar with +

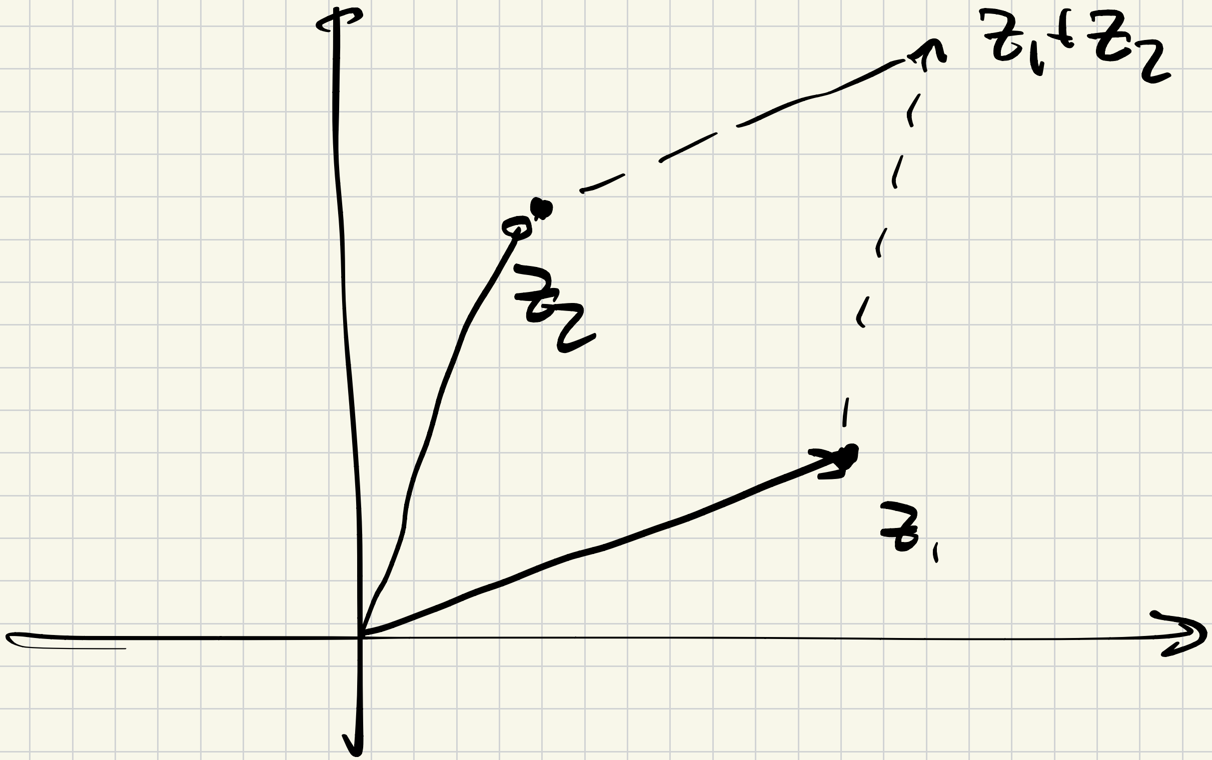
Definition of $+$ and $\times$

$$z_1 = x_1 + i \cdot y_1 \qquad x_1, y_1 \in \mathbb{R}$$

$$z_2 = x_2 + i \cdot y_2 \qquad x_2 \; y_2 \in \mathbb{R}$$

By definition: $\quad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

This corresponds to vector addition on $\mathbb{R}^2$



$\mathbb{C}$

By definition: We use that $i^2 = -1$

$$z_1 \cdot z_2 = (x_1 + i y_1)(x_2 + i y_2) =$$

$$x_1 \cdot x_2 + x_1 \cdot (i y_2) + i y_1 \cdot x_2 + (i y_1)(i y_2)$$

$$= x_1 x_2 + i \cdot x_1 \cdot y_2 + i y_1 \cdot x_2 + \underbrace{i^2 \cdot y_1 \cdot y_2}_{= -y_1 y_2}$$

$$= x_1 x_2 + i \cdot x_1 \cdot y_2 + i\, y_1 \cdot x_2 + \underbrace{i^2 \cdot y_1 \cdot y_2}_{= -y_1 y_2} =$$

$$= \underbrace{\overbrace{\left(x_1 x_2 - y_1 y_2\right)}^{\in \mathbb{R}} + i \cdot \overbrace{\left(x_1 y_2 + y_1 x_2\right)}^{\in \mathbb{R}}}_{\in \mathbb{C}}$$

Definition for $z_1 = x_1 + i y_1$

$z_2 = x_2 + i y_2$

$$z_1 + z_2 := (x_1 + x_2) + (y_1 + y_2) i$$

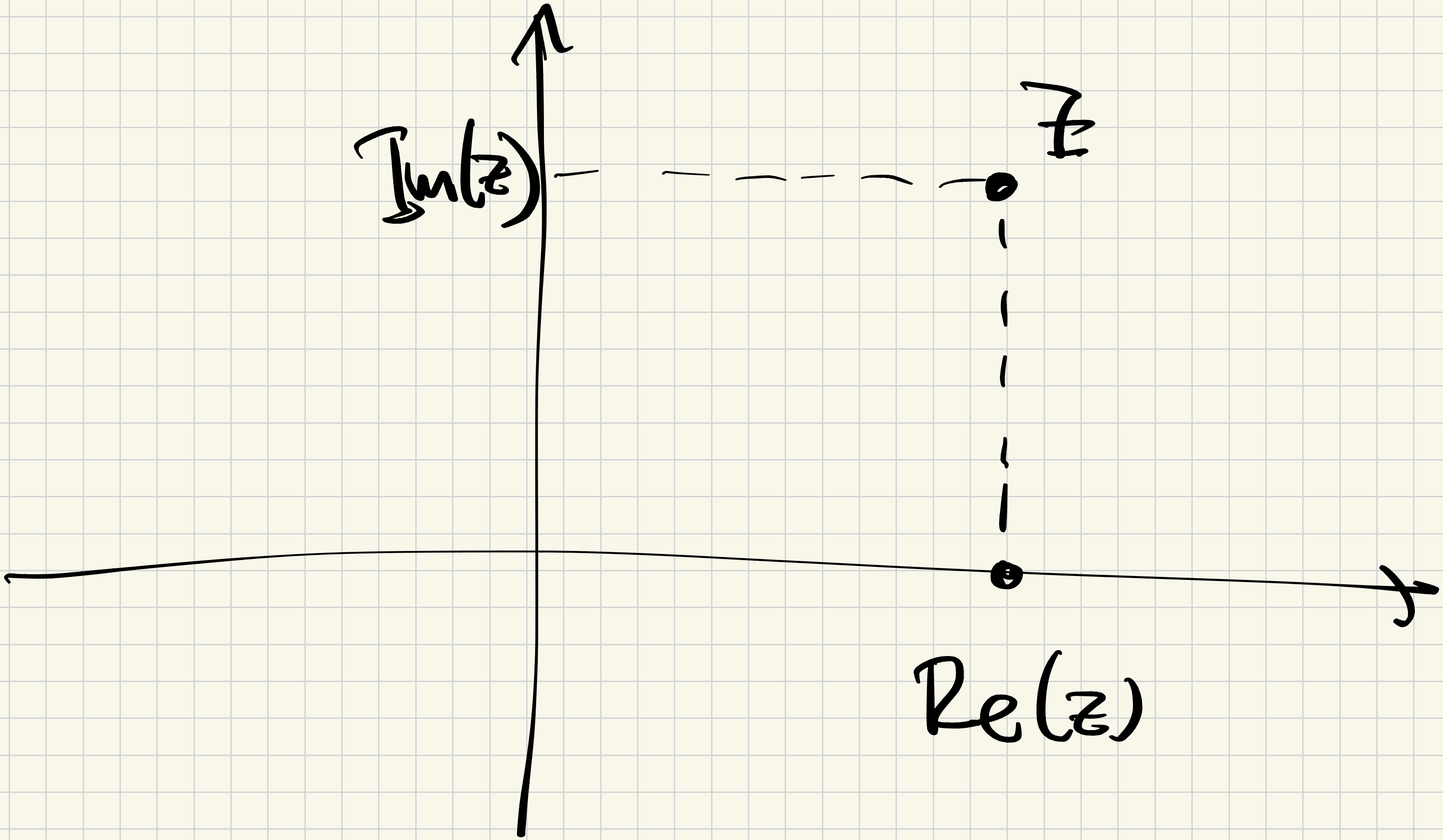
$$z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i$$

Definition For complex number $z = x + iy$ we call

x the real part of z denoted by $\mathrm{Re}(z) = x$

y the imaginary part of z denoted by $\mathrm{Im}(z) = y$

On the $\mathbb{R}^2 = \mathbb{C}$



In particular, $\mathrm{Im}(z), \mathrm{Re}(z) \in \mathbb{R}$.

Theorem the set of complex numbers with operations $+$, $\times$ is a field.

In particular for any $z \neq 0$ there exists the inverse:

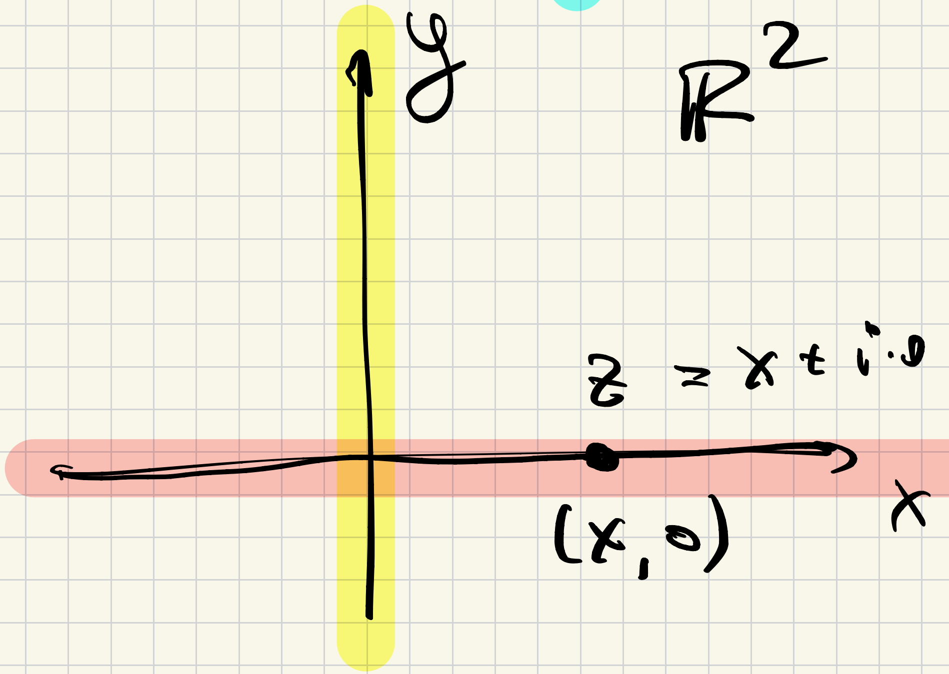
$z^{-1} \in \mathbb{C}$ with $z \cdot z^{-1} = 1$.

Reminder sets of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$$x = x + i \cdot 0$$

Pictorially $\mathbb{R} \subset \mathbb{C}$ is x-axis



Complex numbers on y-axis have form iy are called purely imaginary

Formula for the inverse:

Let $z = x + iy \in \mathbb{C}$, $\neq 0$

then $$z^{-1} = \frac{x - iy}{x^2 + y^2} = $$

$$= \underbrace{\frac{x}{x^2+y^2}}_{\text{Re}(z^{-1})} - \underbrace{\frac{y}{x^2+y^2}}_{\text{Im}(z^{-1})} \cdot i$$

Proof: We need to check $z \cdot z^{-1} = 1$:

$$(x+iy)\cdot\left(\frac{x-iy}{x^2+y^2}\right) =$$

$$i^2\cdot y^2 = -y^2$$

$$= \frac{(x+iy)\cdot(x-iy)}{x^2+y^2} = \frac{x^2-(iy)^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1 \quad \blacksquare$$

$$(a+b)(a-b) = a^2 - b^2$$